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On the asymptotic solution of the Orr–Sommerfeld equation by the method of multiple-scales

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The method of multiple-scales is used to obtain the asymptotic solution of the Orr–Sommerfeld equation. For the special case of a linear velocity profile, the solution so obtained agrees well with an approximation of the exact solution which is known. For the general case, transformations on both the dependent and independent variables are introduced to obtain a zeroth-order equation which differs from the inner equation studied so far. On the ground of the favourable comparison for the special case, the asymptotic solution constructed is expected to be uniformly valid.

1. Introduction

The Orr–Sommerfeld equation governs two-dimensional infinitesimal disturbances in a basic two-dimensional flow. (Three-dimensional disturbances can also be considered using the Squire transformation.) The differential equation is linear, ordinary, of the fourth order, containing three parameters, and has one or more simple turning points. No exact solution of this equation has been obtained for a general velocity profile. The situation of primary physical interest is when one of the parameters is large, and most solutions attempted are constructed for that parameter tending to infinity. Notably, there are two approaches in its solution. In the heuristic approach, two asymptotic expansions: an inner expansion about the turning point and an outer expansion away from it, are considered simultaneously. The difficulty in this approach is that one of the solutions obtained from the outer expansion is singular at the turning point. In the more rigorous method of comparison equations, the asymptotic solutions are constructed in terms of those of a simpler reference equation. In this paper, the asymptotic solution is obtained by the method of multiple-scales; Mahony (1962), Van Dyke (1964). Four linearly independent solutions are constructed from a stretched form of the Orr–Sommerfeld equation, with the unknown function treated as dependent on both the stretched and unstretched variables. By virtue of the expansion procedure, the solutions are expected to be uniformly valid. For the special case when the basic velocity profile is linear, the exact solution has been obtained by Sommerfeld (1908). The asymptotic solution is compared with an appropriate expansion of the exact solution and it is seen that there is good agreement between them.

For a general velocity profile, consideration is limited to the case of a simple turning point. The dependent and independent variables are transformed to new variables which are expected to be more appropriate for the construction of uniformly valid asymptotic solutions. Four solutions of the zeroth-order equation are obtained.

Recently, Graebel (1966) applied the method of matched-expansions to the classical inner and outer solutions to obtain eigenvalue relations. As the multiple-scales method is a variant of the method of matched-expansions, it is readily seen that the zeroth-order equation obtained by the former method is of the same form as the inner equation obtained by the latter, but with ordinary differential operators replaced by the corresponding partial operators. In this manner, its solutions have coefficients which are functions of the unstretched variable. These functions are then chosen to ensure the uniform validity of the asymptotic expansion. Further, in using the multiple-scales method, there is no need to consider any solution which is singular at the turning point.

2. Asymptotic solution of the Orr–Sommerfeld equation for a linear velocity profile

In the non-dimensional form, the Orr–Sommerfeld equation is

$$(D^2-\alpha^2)^2\phi = i\alpha R\{(\bar{u}-c)(D^2-\alpha^2)\phi-(D^2\bar{u})\phi\} \tag{2.1}$$

where $D = d/dx$; α is the wave-number of the disturbance; ϕ is the amplitude of the disturbance; R is the Reynolds number; $\bar{u}$ is the basic velocity profile; and c is a complex number denoting the speed and amplification of the disturbance. Of interest is the eigenvalue relation obtained by imposing the boundary conditions $\phi = D\phi = 0$ at two points to the solution of (2.1). When the basic velocity profile is linear ($\bar{u} = x$), equation (2.1) becomes

$$(D^2-\alpha^2)^2\phi = i\alpha R\{(x-c)(D^2-\alpha^2)\phi\}. \tag{2.2}$$

Here we shall be concerned with the asymptotic solution when αR tends to infinity.

The substitution $z = i(x-c)$, which is a special case of the substitution used in §3, and the stretching $z = (\alpha R)^{-\frac{1}{3}}\xi$ transform (2.2) into

$$\ddot{\ddot{\phi}}+(\xi+2\epsilon^2\alpha^2)\ddot{\phi}+(\epsilon^2\alpha^2\xi+\epsilon^4\alpha^4)\phi = 0, \tag{2.3}$$

where $\epsilon^3 = 1/\alpha R$ and the dot denotes differentiation with respect to ξ. The function ϕ is now treated as a function of both z and ξ, and the ordinary differential operator is replaced by the corresponding partial operator. Hence, we write

$$\left.\begin{aligned}\frac{d^2}{d\xi^2} &= \frac{\partial^2}{\partial\xi^2}+2\epsilon\frac{\partial^2}{\partial z\,\partial\xi}+\epsilon^2\frac{\partial^2}{\partial z^2},\\ \frac{d^4}{d\xi^4} &= \frac{\partial^4}{\partial\xi^4}+4\epsilon\frac{\partial^4}{\partial z\,\partial\xi^3}+6\epsilon^2\frac{\partial^4}{\partial z^2\,\partial\xi^2}+4\epsilon^3\frac{\partial^4}{\partial z^3\,\partial\xi}+\epsilon^4\frac{\partial^4}{\partial z^4}.\end{aligned}\right\} \tag{2.4}$$

Substitution of (2.4) into (2.3) yields

$$[\phi_{,\xi\xi\xi\xi}+\xi\phi_{,\xi\xi}]+\epsilon[4\phi_{,\xi\xi\xi z}+2\xi\phi_{,\xi z}]+\epsilon^2[6\phi_{,\xi\xi zz}+\xi\phi_{,zz}+2\alpha^2\phi_{,\xi\xi}+\alpha^2\xi\phi]$$
$$+\epsilon^3[4\phi_{,\xi zzz}+4\alpha^2\phi_{,\xi z}]+\epsilon^4[\phi_{,zzzz}+2\alpha^2\phi_{,zz}+\alpha^2\phi] = 0. \quad (2.5)$$

We seek an asymptotic expansion for ϕ of the form

$$\phi(z,\xi,\epsilon) = \sum_{n=0}^{\infty} \epsilon^n \phi_n(z,\xi) \quad (\epsilon \to 0). \quad (2.6)$$

On substitution of (2.6) in (2.5) and equating the coefficients of powers of ϵ to zero, an infinite number of equations for the determination of ϕ_n is obtained. With the notation

$$\mathscr{L} \equiv \frac{\partial^4}{\partial \xi^4}+\xi\frac{\partial^2}{\partial \xi^2},$$

the first three of these are

$$\mathscr{L}(\phi_0) = 0, \quad (2.7)$$

$$\mathscr{L}(\phi_1) = -4\phi_{0,\xi\xi\xi z}-2\xi\phi_{0,\xi z} = h_1(z,\xi), \quad (2.8)$$

$$\mathscr{L}(\phi_2) = -4\phi_{1,\xi\xi\xi z}-2\xi\phi_{1,\xi z}-6\phi_{0,\xi\xi zz}-\xi\phi_{0,zz}-2\alpha^2\phi_{0,\xi\xi}-\alpha^2\xi\phi_0 = h_2(z,\xi). \quad (2.9)$$

It is clear that $\phi_n (n \geqslant 1)$ may be obtained in an iterative manner once ϕ_0 is known.

The solutions of (2.7) are integrals of the Airy functions. Alternately, they may be expressed as

$$F_k(\xi) = \int_{c_k} t^{-2}\, e^{\xi t+\frac{1}{3}t^3}\, dt, \quad (2.10)$$

where c_k is such that $[e^{\xi t+\frac{1}{3}t^3}]_{c_k} = 0$. In the unshaded sectors of the t-plane, as shown in figure 1, t^3 has negative real part. The integral in (2.10) converges when c_k is chosen as any of the three contours depicted. Further, the integrand has a pole of order two at the origin with residue ξ, so that if c is chosen to be a closed contour around the origin, say c_0, then

$$F_0 = \int_{c_0} t^{-2}\, e^{\xi t+\frac{1}{3}t^3}\, dt = 2\pi i \xi. \quad (2.11)$$

It is readily checked that $F_k, k=1, 2, 3$, are linearly independent. The four solutions F_k ($k = 0, 1, 2, 3$) corresponding to c_k are not linearly independent as, by Cauchy's theorem,

$$F_1+F_2+F_3 = F_0.$$

A fourth solution is independent of ξ, and may be an arbitrary function of z. Four solutions of ϕ_0 may thus be chosen as $D_0(z)$, $C_0(z)\,\xi$, and any two of F_1, F_2 and F_3, with coefficients which are functions of z.

To construct a uniformly valid asymptotic expansion, the behaviour of $F_k(\xi)$ when $|\xi|$ tends to infinity will be required. This is obtained by the method of steepest descents in appendix A. The result of this evaluation is that the ξ-plane is divided into three equal sectors S_k, $k = 1, 2, 3$, (see figure 2) in each of which $F_k(\xi)$ is exponentially decreasing while the other two functions are exponentially increasing. Let us suppose that the boundary conditions $\phi = D\phi = 0$

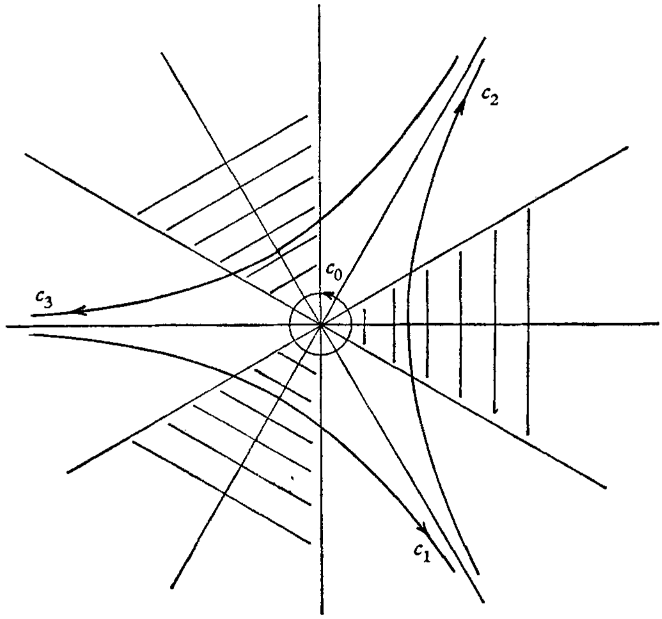

FIGURE 1. Suitable contours in the t-plane.

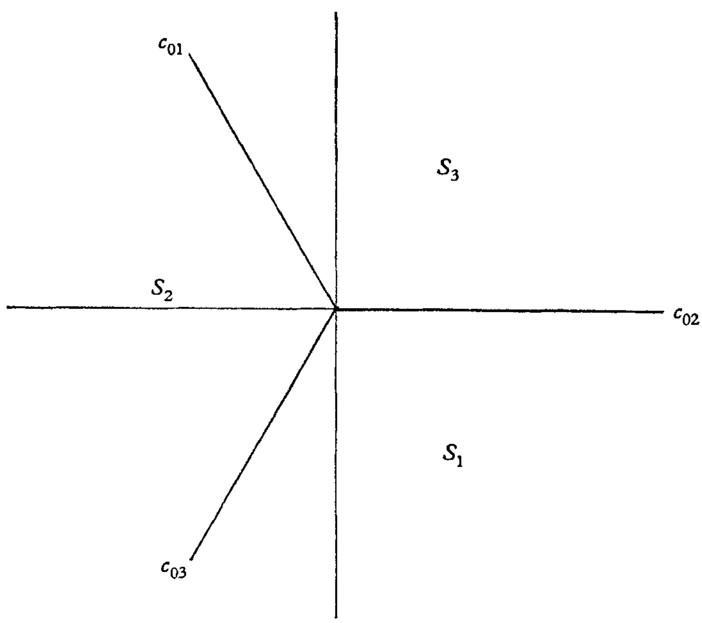

FIGURE 2. Diagram for the choice of branches.

are to be applied at points x_1 and x_2, with corresponding points z_1 and z_2, which lie respectively in the sectors S_1 and S_2. We write the solution of ϕ_0 as

$$\phi_0 = A_0(z)F_1(\xi) + B_0(z)F_2(\xi) + C_0(z)\xi + D_0(z). \tag{2.12}$$

The arbitrary functions of z are to be determined from consideration of higher-order terms.

The derivatives of ϕ_0 will be required in the determination of ϕ_1. It is readily seen that differentiation of $F_k(\xi)$ with respect to ξ under the integral sign is permissible (Sanone & Gerretsen 1960). Hence, we have

$$\frac{d^n F_k}{d\xi^n} = \int_{c_k} t^{n-2}\, e^{\xi t + \frac{1}{3}t^3}\, dt. \tag{2.13}$$

It follows from appendix A that when F_k is exponentially decreasing or increasing, so are its derivatives. From (2.8), we have

$$\mathscr{L}(\phi_1) = -4A_0'(z)F_1'''(\xi) - 2\xi A_0'(z)F_1'(\xi) - 4B_0'(z)F_2'''(\xi) - 2\xi B_0'(z)F_2'(\xi) - 2C_0'(z)\xi = h_1(z,\xi).$$

Here, as in what follows, the prime denotes differentiation with respect to the argument of the function. Using the variation of parameters method, it is readily seen that the general solution of ϕ_1 is

$$\phi_1 = A_1(z)F_1(\xi) + B_1(z)F_2(\xi) + C_1(z)\xi + D_1(z) + F_1(\xi)\int_0^\xi \frac{W_1(\zeta)}{W} h_1(z,\zeta)\,d\zeta$$
$$+ F_2(\xi)\int_0^\xi \frac{W_2(\zeta)}{W} h_1(z,\zeta)\,d\zeta + \xi\int_0^\xi \frac{W_3(\zeta)}{W} h_1(z,\zeta)\,d\zeta + \int_0^\xi \frac{W_4(\zeta)}{W} h_1(z,\zeta)\,d\zeta. \tag{2.14}$$

Here W is the Wronskian of the four solutions F_1, F_2, ξ and 1; W_k ($k = 1, 2, 3, 4$) is the determinant obtained by replacing the kth column of W by the column $(0, 0, 0, 1)$. It follows that

$$\left.\begin{aligned} &W = \text{constant}, \\ &W_1(\zeta) = F_2''(\zeta); \quad W_2(\zeta) = -F_1''(\zeta); \\ &W_3(\zeta) = F_2'(\zeta)F_1''(\zeta) - F_1'(\zeta)F_2''(\zeta); \\ &W_4(\zeta) = -\zeta W_3(\zeta) - F_1(\zeta)F_2''(\zeta) + F_2(\zeta)F_1''(\zeta). \end{aligned}\right\} \tag{2.15}$$

To determine the arbitrary functions of z in ϕ_0, we impose the condition that the terms of the asymptotic expansion (6) satisfy

$$\epsilon\phi_n(z,\xi) = o(\phi_{n-1}(z,\xi)) \quad \left\{\begin{matrix} \epsilon \to 0, \\ \text{uniformly in } z, \end{matrix}\right\} \tag{2.16}$$

and the same shall be true for their partial derivatives.

It is obvious that the complementary function of ϕ_1 will satisfy (2.16), so only the particular integral in (2.14) need be considered. This involves integrals of the product of h_1 and derivatives of F_k. Their asymptotic behaviour may be approximated by substituting for F_k and its derivatives their corresponding asymptotic representations before evaluating the integrals. For the present purpose, an order of magnitude consideration will suffice. Since ϕ_0 is in fact the sum of four solutions of $\mathscr{L}(\phi_0) = 0$, condition (16) will be applied to each of these functions. For example, in comparing $A_0(z)F_1(\xi)$ with

$$F_1(\xi)\int_0^\xi \frac{W_1(\zeta)}{W} h_1(\zeta, z)\,d\zeta,$$

we consider only the contribution of $A_0(z)F_1(\xi)$ to $h_1(z, \xi)$. Further, it is only necessary to carry out the comparison for ξ in either S_1 or S_2 since ϕ_0 behaves similarly in both sectors. It follows from appendix A that, as $|\xi|$ tends to infinity in S_1,

$$\left.\begin{aligned} &W_1(\xi) = O(\xi^{-\frac{1}{4}}\exp\{\tfrac{2}{3}|\xi|^{\frac{3}{2}}\}), \quad W_2(\xi) = O(\xi^{-\frac{1}{4}}\exp\{-\tfrac{2}{3}|\xi|^{\frac{3}{2}}\}),\\ &W_3(\xi) = O(\xi^{-1}); \quad W_4(\xi) = O(1);\\ &-4A_0'(z)F_1'''(\xi) - 2\xi A_0'(z)F_1'(\xi) = O(\xi^{\frac{1}{4}}\exp\{-\tfrac{2}{3}|\xi|^{\frac{3}{2}}\}),\\ &-4B_0'(z)F'''(\xi) - 2\xi B_0'(z)F'(\xi) = O(\xi^{\frac{1}{4}}\exp\{\tfrac{2}{3}|\xi|^{\frac{3}{2}}\}), \quad -2\xi C'(z) = O(\xi).\end{aligned}\right\} \tag{2.17}$$

Condition (2.16) is satisfied if

$$\epsilon\int_0^\xi \frac{W_k(\zeta)}{W} h_1(z, \zeta)\,d\zeta = o(1)$$

holds for $|\xi|$ tending to infinity. It is readily seen that this requires

$$A_0'(z) = B_0'(z) = C_0'(z) = 0 \tag{2.18}$$

and that no information is available to determine $D_0(z)$. We denote the constants obtained from (18) by A_{00}, B_{00}, C_{00}, and proceed to consider ϕ_2, with the knowledge that

$$\left.\begin{aligned} \phi_0 &= A_{00}F_1(\xi) + B_{00}F_2(\xi) + C_{00}\xi + D_0(z),\\ \phi_1 &= A_1(z)F_1(\xi) + B_1(z)F_2(\xi) + C_1(z)\xi + D_1(z).\end{aligned}\right\} \tag{2.19}$$

Similar to (2.14), the complete solution for ϕ_2 may be written down immediately. The nonhomogeneous part of equation (2.9) giving rise to the particular integral is

$$h_2(z, \xi) = -4\phi_{1,\xi\xi\xi z} - 2\xi\phi_{1,\xi z} - 6\phi_{0,\xi\xi zz} - \xi\phi_{0,zz} - 2\alpha^2\phi_{0,\xi\xi} - \alpha^2\xi\phi_0.$$

As in (2.18), we require

$$A_1'(z) = B_1'(z) = C_1'(z) = 0,$$

reducing h_2 to
$$h_2(z, \xi) = -\xi\phi_{0,zz} - 2\alpha^2\phi_{0,\xi\xi} - \alpha^2\xi\phi_0.$$

Now, we have

$$\begin{aligned} &-2\alpha^2A_{00}F_1''(\xi) - \alpha^2\xi A_{00}F_1(\xi) = O(\xi^{-\frac{1}{4}}\exp\{-\tfrac{2}{3}|\xi|^{\frac{3}{2}}\}),\\ &-2\alpha^2B_{00}F_2''(\xi) - \alpha^2\xi B_{00}F_2(\xi) = O(\xi^{-\frac{1}{4}}\exp\{\tfrac{2}{3}|\xi|^{\frac{3}{2}}\}),\\ &-\alpha^2\xi C_{00}\xi = -\alpha^2C_{00}\xi^2,\\ &-\xi D_0(z)_{,zz} - \alpha^2D_0(z) = -\xi[D_0(z)_{,zz} + \alpha^2D_0(z)].\end{aligned}$$

In carrying out the comparison of the individual functions, we see that

$$\epsilon\int_0^\xi \frac{W_3(\zeta)}{W}[-\alpha^2C_{00}\zeta^2]\,d\zeta \neq O(1),$$

$$\epsilon\int_0^\xi \frac{W_4(\zeta)}{W}[-\xi(D_0(z)_{,zz} + \alpha^2D_0(z))]\,d\zeta \neq O(1).$$

Hence, we require $$C_{00} = 0,$$

and $$D_0(z)_{,zz} + \alpha^2 D_0(z) = 0$$

yielding $$D_0(z) = D_{01}\, e^{i\alpha z} + D_{02}\, e^{-i\alpha z}. \tag{2.20}$$

The zeroth-order solution ϕ_0 is therefore completely determined as

$$\phi_0(z, \xi) = A_{00} F_1(\xi) + B_{00} F_2(\xi) + D_{01}\, e^{i\alpha z} + D_{02}\, e^{-i\alpha z}. \tag{2.21}$$

For our present comparison purpose, the higher-order terms need not be considered. It is shown in appendix B that there is good agreement between this zeroth-order solution and an appropriate expansion of the exact solution.

3. Asymptotic solution of the Orr–Sommerfeld equation for a general velocity profile

In studying the Orr–Sommerfeld equation (2.1) with a general velocity profile, it is assumed in the following that $\bar{u}$ is a monotonic function of x, so that $(\bar{u} - c)$ has a simple zero.

We introduce the following transformations on both the dependent and independent variables as in Lin & Rabenstein (1960):

$$z = \left[\frac{3}{2}\int_{x_0}^{x} P_0^{\frac{1}{2}}\, dx\right]^{\frac{2}{3}}, \quad \psi = \phi\left(\frac{P_0}{z}\right)^{\frac{3}{4}}, \tag{3.1}$$

where $P_0 = -i(\bar{u} - c)$ and x_0 is the turning point where $P_0(x_0) = 0$. For the special case $\bar{u} = x$, these reduce essentially to the transformation used in §2. Here, the choice of the independent variable is based on a comparison of two classical asymptotic solutions, which in essence are inner and outer solutions. The transformation of the dependent variable serves only to simplify the subsequent equation. From heuristic arguments, it is expected that this independent variable is the appropriate choice for the construction of uniformly valid asymptotic solutions, (Lin 1955, p. 128). In terms of these variables, the Orr–Sommerfeld equation (2.1) becomes

$$\epsilon^2 \psi^{\text{iv}} + (z + \epsilon^3 p)\psi'' + (q_0 + \epsilon^3 q_1)\psi' + (r_0 + \epsilon^3 r_1)\psi = 0, \tag{3.2}$$

where p, q_0, q_1, r_0 and r_1 are functions of z:

$$\left.\begin{aligned} q_0 &= 1 + \tfrac{1}{2} P_0^{-\frac{6}{4}} z^{\frac{6}{4}} - \tfrac{3}{2} P_0^{-\frac{6}{4}} z^{\frac{6}{4}} \frac{dP_0}{dx}, \\ r_0 &= -\tfrac{3}{4} P_0^{-\frac{3}{2}} z^{\frac{1}{2}} \frac{dP_0}{dx} - \tfrac{9}{16} z^{-1} + \tfrac{21}{16} P_0^{-3} z^2 \left(\frac{dP_0}{dx}\right)^2 \\ &\quad - \tfrac{3}{4} P_0^{-2} z^2 \frac{d^2 P_0}{dx^2} + \left(-\alpha^2 P_0 + i\frac{d^2\bar{u}}{dx^2}\right) P_0^{-2} z^2. \end{aligned}\right\} \tag{3.3}$$

The other functions of z are not required for the purpose of obtaining the zeroth order asymptotic solution, and so will not be presented. For the sake of brevity, P_0 is retained as a function of x. Clearly, if $\bar{u}$ is given, it is possible to express x in terms of z.

If we introduce the stretching transformation $z = \epsilon\xi$, and write

$$\frac{d}{dz} = \frac{\partial}{\partial z} + \epsilon^{-1}\frac{\partial}{\partial \xi},$$

equation (3.2) becomes

$$\begin{aligned}\psi_{,\xi\xi\xi\xi} + \xi\psi_{,\xi\xi} + q_0\psi_{,\xi} &+ \epsilon\{4\psi_{,\xi\xi\xi z} + 2\xi\psi_{,\xi z} + q_0\psi_{,z} + r_0\psi\}\\ &+ \epsilon^2\{6\psi_{,\xi\xi zz} + \xi\psi_{,zz} + p\psi_{,\xi\xi}\}\\ &+ \epsilon^3\{4\psi_{,\xi zzz} + q_1\psi_{,\xi} + 2p\psi_{,\xi z}\}\\ &+ \epsilon^4\{\psi_{,zzzz} + r_1\psi + p\psi_{,zz} + q_1\psi_{,z}\} = 0. \qquad (3.4)\end{aligned}$$

The zeroth-order equation is

$$\psi_{0,\xi\xi\xi\xi} + \xi\psi_{0,\xi\xi} + q_0\psi_{0,\xi} = 0. \qquad (3.5)$$

It differs from the inner equation of the form $\psi^{\text{iv}} + \xi\phi'' = 0$ used so far in the study of the Orr–Sommerfeld equation. On the strength of the comparison carried out in appendix B for the special case $\bar{u} = x$, it seems reasonable to expect that the solutions of (3.5) will be a good approximation to those of the original equation for large values of αR.

With $v = \psi_{0,\xi}$, equation (3.5) becomes

$$v_{,\xi\xi\xi} + \xi v_{,\xi} + q_0 v = 0. \qquad (3.6)$$

In solving this equation, $q_0(z)$ may be treated as a constant since all partial differentiations are with respect to ξ. However, to simplify the subsequent comparison necessary for the complete determination of ψ_0, we replace it by the constant q_{00} obtained by making the following approximations:

$$P_0 = -i(\bar{u}-c) \doteq -i\bar{u}'(x_0)(x-x_0), \quad P' \doteq -i\bar{u}'(x_0), \quad z \doteq [-i\bar{u}'(x_0)]^{\frac{1}{3}}(x-x_0).$$

Hence
$$q_0 \doteq -\frac{1}{2}\left(1-\frac{i}{\bar{u}'(x_0)}\right) = q_{00}$$

is a constant. This simplification may be considered as the result of expanding $q_0(z)$ in a Taylor series and writing z^n as $\epsilon^n\xi^n$. Then q_{00} is formally of order one, and all succeeding terms are formally of order ϵ or smaller. Instead of (3.6), we consider

$$v_{,\xi\xi\xi} + \xi v_{,\xi} + q_{00} v = 0. \qquad (3.7)$$

Langer (1955) and Hershenov (1957) have considered the asymptotic solution of

$$v''' + \lambda^2 z v' + 3\lambda^2\mu v = 0, \qquad (3.8)$$

where the prime denotes differentiation with respect to z and μ is a complex constant. If we make the transformation $z = \lambda^{-\frac{2}{3}}\xi$, equation (3.8) becomes

$$\dddot{v} + \xi\dot{v} + 3\mu v = 0.$$

Hence, the solution obtained by these authors is directly applicable.

It is readily seen that the solutions of (3.7) can be obtained in integral form as

$$I_k(\xi) = \int_{C_k} t^{q_{00}-1}\exp\{\xi t + \tfrac{1}{3}t^3\}\,dt \quad (k = 1, 2, 3), \qquad (3.9)$$

where c_k is such that $[t^{q_{00}}\exp\{\xi t+\frac{1}{3}t^3\}]_{c_k}=0$. The solutions of (3.5), with q_{00} replacing q_0, are therefore

$$F_k(\xi)=\int v\,d\xi+\text{constant}$$

$$=\int_{c_k} t^{q_{00}-2}\exp\{\xi t+\tfrac{1}{3}t^3\}dt+\text{constant}. \tag{3.10}$$

Hence, knowing the solutions of (3.7), we can obtain those of (3.5) by replacing q_{00} by $q_{00}-1$. Similarly, the nth order derivative of I_k can be obtained by replacing q_{00} by $q_{00}+n$.

The asymptotic expansion of (3.10) is as follows: (Hershenov 1957)

$$I_k(\xi)\sim a_k\pi^{\frac{1}{2}}\exp\{-\tfrac{3}{2}\pi(\tfrac{1}{3}q_{00}+\tfrac{1}{2})\}\xi^{(\frac{1}{2}q_{00}-\frac{3}{4})}\exp(-\tfrac{2}{3}i\xi^{\frac{3}{2}})[1+O(1/|\xi|)] \tag{3.11}$$

valid for $$-\frac{4\pi}{3}-\frac{8\pi}{3}(k-2)<\arg\xi<\frac{2\pi}{3}-\frac{8\pi}{3}(k-2),\ \text{mod}\,4\pi,$$

where $k=1,2,3$; and $a_1=-1$, $a_2=-1$, $a_3=1$. As in the case considered in §2, the ξ-plane is divided into three equal sectors in each of which there is one solution which is exponentially decreasing as $|\xi|$ tends to infinity and two solutions which are exponentially increasing. The tentative solution of ψ_0 is therefore

$$\psi_0=A_0(z)F_1(\xi)+B_0(z)F_2(\xi)+C_0(z)F_3(\xi)+D_0(z), \tag{3.12}$$

where the functions of z are to be determined. Proceeding as in §2 and using the same notations, we require that as $|\xi|$ tends to infinity, h_1 tends to zero. This yields the following first order differential equations for the determination of $A_0(z)$, $B_0(z)$, $C_0(z)$ and $D_0(z)$:

$$(K\xi^{\frac{3}{2}}+q_0)A_0(z)_{,z}+r_0A_0(z)=0, \tag{3.13}$$

where $$K=2\,e^{-\frac{1}{2}\pi}[2\,e^{-\pi}+1].$$

Similar equations govern $B_0(z)$ and $C_0(z)$; and

$$q_0D_0(z)_{,z}+r_0D_0(z)=0. \tag{3.14}$$

Since we suppose these functions depend on z only, $\xi^{\frac{3}{2}}$ must be expressed as $\epsilon^{-\frac{3}{2}}z^{\frac{3}{2}}$. The solutions of the above equations are

$$\left.\begin{aligned}A_0(z)&=A_{00}\exp\left\{-\epsilon^{\frac{3}{2}}\int_0^z\frac{r_0}{Kz^{\frac{3}{2}}+\epsilon^{\frac{3}{2}}q_0}dz\right.\\&=A_{00}Z(z),\\B_0(z)&=B_{00}Z(z),\\C_0(z)&=C_{00}Z(z),\\D_0(z)&=D_{00}\exp\left\{-\int_0^z\frac{r_0}{q_0}dz\right\}=D_{00}Y_0(z).\end{aligned}\right\} \tag{3.15}$$

Hence, we have the complete zeroth order solution

$$\psi_0(z,\xi)=A_{00}Z(z)F_1(z)+B_{00}Z(z)F_2(z)+C_{00}Z(z)F_3(\xi)+D_{00}Y_0(z). \tag{3.16}$$

4. Concluding remarks

A multiple-scales method is used to construct uniformly valid asymptotic solutions for the Orr–Sommerfeld equation. For the case of a linear velocity profile, the solution constructed agrees with an expansion of the exact solution obtained by Sommerfeld. The solution obtained for a general velocity profile is being applied to the case of plane Poiseuille flow to obtain the characteristic equation. The calculation will be published when completed.

We observe that in using the multiple-scales method, one attempts to construct uniformly valid solutions from the inner equation, formulated as dependent on both the stretched and unstretched variables. The inviscid equation, obtained by setting $\epsilon = 0$ in the unstretched equation, is not considered. Hence there is no need to use any solution which is singular at the turning point.

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Appendix A. The asymptotic solution of $\phi^{\mathrm{iv}} + \xi\phi'' = 0$

It is readily seen that the equation admits solutions

$$F_k(\xi) = \int_{c_k} e^{\xi t} f(t)\,dt, \tag{A 1}$$

where t is a complex variable, $f(t) = t^{-2} e^{\frac{1}{3}t^3}$, and c_k is a contour such that $[e^{\xi t + \frac{1}{3}t^3}]_{c_k} = 0$. The integral converges when c_k is chosen as in figure 1. Denoting F_k by $F_k(\xi, -2)$, where -2 denotes the power of t in $f(t)$, differentiation with respect to ξ yields

$$\frac{d^n F_k}{d\xi^n}(\xi, -2) = F_k(\xi, n-2). \tag{A 2}$$

The behaviour of $F_k(\xi, \lambda)$, where λ is an integer, for $|\xi| \to \infty$ may be evaluated by the method of steepest descents. The following treatment follows closely that of Rabenstein (1958).

Let $t = \sigma\tau = (-i\xi^{\frac{1}{2}})\tau$. Writing $\eta = i\frac{2}{3}\xi^{\frac{3}{2}}$, we have

$$\tfrac{1}{3}t^3 + \xi t = \eta(\tfrac{1}{2}\tau^3 - \tfrac{3}{2}\tau) = |\eta| g(\tau),$$

where
$$g(\tau) = e^{iv}(\tfrac{1}{2}\tau^3 - \tfrac{3}{2}\tau)$$

and
$$v = \arg\eta = \tfrac{3}{2}\arg\xi + \tfrac{1}{2}\pi.$$

The integral (A 1) is then transformed into

$$F_k = (-i\xi^{\frac{1}{2}})^{\lambda+1} \int_{c'} \tau^{\lambda} e^{|\eta| g(\tau)}\,d\tau, \tag{A 3}$$

where c'_k is the image of c_k in the τ-plane. Clearly, the location of c'_k depends on the choice of a branch of $\sigma = -(\frac{3}{2}\eta)^{\frac{1}{3}} i^{\frac{2}{3}}$.

For the integral (A 3) to converge, $e^{iv}\tau^3$ must have negative real part, as $|\tau| \to \infty$ for fixed $\arg \tau = \psi$, which requires

$$\tfrac{1}{2}\pi + 2n\pi < v + 3\psi < \tfrac{3}{2}\pi + 2n\pi. \tag{A 4}$$

The function $g(\tau)$ has a saddle-point at $\tau = 1$ where $g'(\tau) = 0$. The path of steepest descent through $\tau = 1$ is given by

$$\operatorname{Im}\,[g(\tau) - g(1)] = 0. \tag{A 5}$$

To plot this path, let $\tau - 1 = \rho\, e^{i\omega}$, then

$$g(\tau) - g(1) = e^{iv}[\tfrac{3}{2}\rho^2\, e^{i2\omega} + \tfrac{1}{2}\rho^3\, e^{i3\omega}]$$

and (A 5) becomes

$$\rho^2[3 \sin(v + 2\omega) + \rho \sin(v + 3\omega)] = 0,$$

$$\frac{1}{\rho} = -\frac{1}{3}\frac{\sin(v + 3\omega)}{\sin(v + 2\omega)}.$$

As $\rho \to \infty$, ω approaches ψ and we require $\sin(v + 3\omega) \doteq 0$. This, together with (A 4), is satisfied if $\psi = \pm\tfrac{1}{3}\pi - \tfrac{1}{3}v - 2n\pi$.

The idea now is to choose for each F_k a branch of $\sigma = -(\tfrac{3}{2}\eta)^{\frac{1}{3}} i^{\frac{2}{3}}$ so that for v in each of the regions

$$-\pi + 2n\pi < v < \pi + 2n\pi,$$

c_k' goes to infinity in the same two sectors of the τ-plane as the path of steepest descent. Then c_k' can be deformed to coincide with the steepest path. This requirement will be satisfied if we take $n = k - 2$ and take $\sigma = |\tfrac{3}{2}\eta|^{\frac{1}{3}}\, e^{i\frac{1}{3}v}$ for v in the range

$$-\pi + 2\pi(k-2) < v < \pi + 2\pi(k-2).$$

To see that this choice is correct, let c_k go to infinity in the t-plane with directions

$$\arg t = \pm\tfrac{1}{3}\pi + \tfrac{2}{3}\pi(k-2),$$

then since $\arg \tau = \arg t - \arg \sigma$, we see that c_k' goes to infinity in the τ-plane with directions

$$\arg \tau = \pm\tfrac{1}{3}\pi - \tfrac{1}{3}\{v - 2\pi(k-2)\}.$$

Therefore, with $n = k - 2$, the requirement we set out to achieve is satisfied.

To evaluate F_k by the method of steepest descents, we set

$$g(\tau) - g(1) = e^{iv}[\tfrac{3}{2}(\tau - 1)^2 + \tfrac{1}{2}(\tau - 1)^3].$$
$$= -\tfrac{1}{2}s^2$$

If we choose

$$s = (-1)^{k+1} i\, e^{\frac{1}{2}iv}(\tau - 1)(\tau + 2)^{\frac{1}{2}}$$

with the branch cut along the negative real axis from $\tau = -2$ to $\tau = -\infty$ and take the branch of the square root which is real and positive for τ real and $\tau > -2$, then as τ traverses the path of steepest descent in the direction corresponding to c_k', s increases through real values. For example, when $k = 2$, $-\pi < v < \pi$, both c_k' and the steepest path go to infinity with directions $\pm\tfrac{1}{3}\pi - \tfrac{1}{3}v$ in the τ-plane. As $|\tau|$ tends to infinity, $\arg(\tau - 1)$ and $\arg(\tau + 2)$ tend to $\arg \tau$. It is readily seen that along the ray $\arg \tau = -\tfrac{1}{3}\pi - \tfrac{1}{3}v$, $s = -|(\tau - 1)(\tau + 2)^{\frac{1}{2}}|$, while along the ray $\arg \tau = \tfrac{1}{3}\pi - \tfrac{1}{3}v$, $s = |(\tau - 1)(\tau + 2)^{\frac{1}{2}}|$. The derivative $ds/d\tau$ is finite and non-zero in the cut plane, so that near $s = 0$, τ is an analytic function of s. For real s, $|s| \geqslant \delta > 0$, we have

$$|\tau^{\lambda}| < K_1 s^{2\lambda/3}; \quad \left|\frac{d\tau}{ds}\right| \leqslant K_2,$$

where K_1 and K_2 are positive constants.

Hence there exist positive constants K and a such that

$$|\tau^{\lambda}|\left|\frac{d\tau}{ds}\right| \leqslant K\, e^{(\frac{1}{2})as^2}.$$

If we expand $\tau^{\lambda}(d\tau/ds)$ in a Taylor series about $s = 0$, it follows from Watson's lemma, (Sansone & Gerretsen 1960, p. 458) that the series obtained formally by termwise integration over $(-\infty, \infty)$ represents F_k asymptotically for

$$-\tfrac{3}{2}\pi + 2\pi(k-2) < v < \tfrac{3}{2}\pi + 2\pi(k-2).$$

Watson's lemma extends the range of v by $\frac{1}{2}\pi$ in each direction. Noting that $\xi = e^{i\pi}\, e^{i\frac{2}{3}v}|\frac{3}{2}\eta|^{\frac{2}{3}}$, the range of $\arg\xi$ is therefore

$$\tfrac{4}{3}\pi(k-2) < \arg\xi < 2\pi + \tfrac{4}{3}\pi(k-2).$$

The result of the integration is

$$F_k = i\sqrt{\pi}\,(-1)^k e^{-\eta}\,\eta^{-\frac{1}{2}}(\tfrac{3}{2}\eta)^{\frac{1}{3}(\lambda+1)}\left[1 + O\left(\frac{1}{|\eta|}\right)\right] \tag{A 6}$$

$$= i\sqrt{\pi}\,(-1)^k e^{-i\frac{2}{3}\xi^{\frac{3}{2}}}\,(i\tfrac{2}{3}\xi^{\frac{3}{2}})^{-\frac{1}{2}}\,(i\xi^{\frac{3}{2}})^{\frac{1}{3}(\lambda+1)}\left[1 + O\left(\frac{1}{|\xi|^{\frac{3}{2}}}\right)\right].$$

In the ξ-plane, $\mathrm{Re}(\eta) = 0$ on three rays, denoted by c_{0k} $(k = 1, 2, 3)$ extending from the origin: (figure 2)

$$c_{0k}\colon \arg\xi = \tfrac{2}{3}\pi - (k-1)\tfrac{2}{3}\pi.$$

These divide the ξ-plane (a finite neighbourhood S of $z = 0$ since $z = \epsilon\xi$) into three sectors S_k of angle $\frac{2}{3}\pi$. Taking S_k to be the closed sector opposite c_{0k}, it can be seen that the choice of a branch of η is equivalent to restricting z to $S - c_{0k}$ and choosing the branch of η for which $\mathrm{Re}(\eta) \geqslant 0$ in S_k and $\mathrm{Re}(\eta) \leqslant 0$ in the other two sectors.

If we examine the asymptotic relation (A 6), we see that for ξ in any closed subset of the interior of S_k, F_k is exponentially decreasing as $|\xi| \to \infty$, and is exponentially increasing in a closed subset of the interior of either of the other two sectors.

Appendix B. An expansion of the exact solution

Equation (2.2)

$$(D^2 - \alpha^2)^2\phi = i\alpha R\{(x-c)(D^2-\alpha^2)\phi\} \tag{B 1}$$

was solved by Sommerfeld (1908). Using the substitution

$$(D^2 - \alpha^2)\phi = \psi \tag{B 2}$$

equation (B 1) becomes

$$(D^2 - i\alpha R(x-c) + \alpha^2)\psi = 0. \tag{B 3}$$

Transforming the independent variable from x to y by

$$y = \frac{\alpha^2}{(\alpha R)^{\frac{2}{3}}} + i(\alpha R)^{\frac{1}{3}}(x-c), \tag{B 4}$$

equation (B 3) becomes

$$d^2\psi/dy^2 + y\psi = 0, \tag{B 5}$$

the solutions of which are the Airy functions. Instead of using the functions $A_i(-y)$ and $B_i(-y)$, it is more convenient for our present purpose to use the integral

$$\psi_k(y) = \int_{c_k} e^{\frac{1}{3}yt+t^3}\,dt, \tag{B 6}$$

where c_k $(k = 1, 2, 3)$ is as depicted in figure 1. The three solutions are not linearly independent as it follows from Cauchy's theorem that

$$\psi_1 + \psi_2 + \psi_3 = 0.$$

Any two of these three functions form a linearly independent set. The solution for ϕ obtained by the method of variation of constants is

$$\phi(y) = \frac{(\alpha R)^{\frac{1}{3}}}{\alpha}\int_0^y \sin\frac{\alpha}{(\alpha R)^{\frac{1}{3}}}(y-\zeta)\,G(\zeta)\,d\zeta + E_1 e^{\frac{i\alpha}{(\alpha R)^{\frac{1}{3}}}y} + E_2 e^{\frac{-i\alpha}{(\alpha R)^{\frac{1}{3}}}y}, \tag{B 7}$$

where $G(y)$ is a linear combination of any two of the functions $\psi_k(y)$. For our present purpose, we choose $k = 1, 2$.

To compare (B 7) with the asymptotic solution (2.21) we proceed as follows: in § 2, the variables z and ξ were introduced. These are related to y by

$$y = \frac{\alpha^2}{(\alpha R)^{\frac{2}{3}}} + \xi, \quad \frac{i\alpha}{(\alpha R)^{\frac{1}{3}}}y = \frac{i\alpha^3}{\alpha R} + i\alpha z.$$

Therefore, as $\alpha R \to \infty$, we have

$$y \doteq \xi; \quad \frac{i\alpha}{(\alpha R)^{\frac{1}{3}}}y \doteq i\alpha z.$$

The solution for ϕ may be expressed as

$$\phi = \frac{(\alpha R)^{\frac{1}{3}}}{\alpha}\int_0^\xi \sin\frac{\alpha(\xi-\zeta)}{(\alpha R)^{\frac{1}{3}}}\,G(\zeta)\,d\zeta + E_1 e^{i\alpha z} + E_2 e^{-i\alpha z} + O(\epsilon). \tag{B 8}$$

Here, the solutions $e^{\pm i\alpha z}$ agree with those obtained by the method of multiple-scales. As for the integral in (B 8) we can let $(\alpha R)^{\frac{1}{3}}/\alpha$ be absorbed in the constants in $G(\zeta)$, and consider the integral

$$I_k = \int_0^\xi \sin\frac{\alpha(\xi-\zeta)}{(\alpha R)^{\frac{1}{3}}}\,\psi_k(\zeta)\,d\zeta.$$

Replacing $\psi_k(\zeta)$ by its integral representation and inverting the order of integration, which is permissible here (Sansone & Gerretsen 1960, p. 97) we obtain by integrating by parts

$$\begin{aligned} I_k &= \int_{c_k} e^{\frac{1}{3}t^3}\,dt \int_0^\xi e^{\zeta t}\sin\frac{\alpha(\xi-\zeta)}{(\alpha R)^{\frac{1}{3}}}\,d\zeta \\ &= \int_{c_k} \frac{e^{\frac{1}{3}t^3}}{1+1/[t^2(\alpha R)^{\frac{2}{3}}]}\left\{\frac{1}{t}\sin\frac{\alpha\xi}{(\alpha R)^{\frac{1}{3}}} - \frac{1}{t^2(\alpha R)^{\frac{2}{3}}}\cos\frac{\alpha\xi}{(\alpha R)^{\frac{1}{3}}} + \frac{e^{\xi t}}{t^2(\alpha R)^{\frac{1}{3}}}\right\}dt. \end{aligned}$$

If we restrict t to $|t| > 0$, then I_k may be expressed as

$$I_k \doteq \int_{c_k} E_3 t^{-2} e^{\xi t + \frac{1}{3}t^3} dt + E_4 \sin \frac{\alpha\xi}{(\alpha R)^{\frac{1}{3}}} + E_5 \cos \frac{\alpha\xi}{(\alpha R)^{\frac{1}{3}}}.$$

Noting that $\alpha\xi/(\alpha R)^{\frac{1}{3}} = \alpha z$, it is clear that ϕ may be represented asymptotically by

$$\phi = K_1 \int_{c_1} t^{-2} e^{\xi t + \frac{1}{3}t^3} dt + K_2 \int_{c_2} t^{-2} e^{\xi t + \frac{1}{3}t^3} dt + K_3 e^{i\alpha z} + K_4 e^{-i\alpha z}. \tag{B 9}$$

The agreement between this solution and that obtained by the method of multiple-scales is evident.

REFERENCES

GRAEBEL, W. P. 1966 *J. Fluid Mech.* **24**, 497.

HERSHENOV, J. 1957 Master's thesis, Dept. of Math. M.I.T.

LANGER, R. E. 1955 *Duke Math. J.* **22**, 525.

LIN, C. C. 1955 *The Theory of Hydrodynamic Stability*. Cambridge University Press.

LIN, C. C. & RABENSTEIN, A. L. 1960 *Tran. Am. Math. Soc.* **94**, 24.

MAHONY, J. J. 1962 *J. Aust. Math. Soc.* **2**, 440.

RABENSTEIN, A. L. 1958 *Arch. Rat. Mech. Anal.* **1**, 418.

SANSONE, G. & GERRETSEN, J. 1960 *Lectures on the Theory of Functions of a Complex Variable*. Groningen: Noordhoff.

SOMMERFELD, A. 1908 *Atti. del IV Congr. Intern. dei Matematici, Roma*, **3**, 116.

VAN DYKE, M. 1964 *Perturbation Methods in Fluid Mechanics*. New York: Academic Press.